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Probabilistic Performance-Based Earthquake Engineering: A Review

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Abstract: The next generation seismic design and assessment procedures for buildings within the performance-based framework are a radical departure from traditional seismic design practice and performance assessment. The uncertainty and randomness in the building performance and seismic hazard will be captured and quantified in each steps in design and assessment procedure, finally the performance will be measured in terms of direct and indirect economic losses and casualties. The quantification and propagation of uncertainty in every step in the procedure requires robust probabilistic methods that have been developed over the last two decades. This paper summarises the research undertaken to develop the probabilistic performance-based seismic design and assessment procedures for buildings. The analysis methods, fragility functions and seismic hazard qualification, which are key elements in procedures, are defined and discussed.

Keywords: building, intensity measure, incremental dynamic analysis, uncertainty

1. Introduction

The performance-based seismic design and assessment approach, in which the building is expected to satisfy certain performance requirements in its lifetime, make a paradigm shift from traditional design and assessment practice (Franchin, [12]). This approach allows to explicitly considering the uncertainties associated with earthquake loading, structural modelling, and structural response prediction etc. The performance-based design formulations against seismic actions specify number of performance levels that must not be exceeded under seismic actions characterised in terms of mean return periods (*fib*, [13]). The mean return periods of seismic actions can be derived and quantified through probabilistic considerations.

Numerous amount of research work has been conducted over the last two decades to develop the current state of the performance-based design procedures that are intelligently conceived and well tested. However, the procedure has serious limitation in the case of an assessment of existing buildings, where the performance requirement cannot be set on the basis of structural response without considering the damage to non-structural components as well as repair costs. In this case, the determination of performance requirements needs additional uncertain data from several sources, which makes the probabilistic approach unavoidable. On this regards, the reliability analysis to seismic design becomes an effective tool that can be used with moderated level of additional effort. As stated in *fib* [13]), "the mandatory adaptation of probabilistic performance-based design (PBD) codes may be still far away from practice, however, this time lag, should be regarded as an opportunity to familiarise with the approaches before actual application".

Thus, this paper briefly summarises the reported works on PBD and mainly provides information on main elements associated with PBD. The research on PBD can be grouped into: (1) probabilistic methods in earthquake engineering that includes fragility curves, collapse risk assessment, seismic hazard analysis with efficient simulation methods, and structural response prediction etc; and (2) seismic performance assessment of existing building that includes the treatment of epistemic uncertainty associated with structural properties and performance prediction models and effect of analysis methods on the performance prediction. However, this paper focuses on the first group of PBD. The research carried out during the last few years by the author and many co-workers also falls largely within this broad framework. Those has been briefly summarised in this paper.

2. Reliability concepts in seismic design

Der Kiureghian [11] and Pinto [19] summarise the early application on reliability concepts in seismic design. The approach can be grouped into three major categories: (1) based on the theory of random vibration with a particular attention to Rice expression for the mean rate of outcrossing of a scalar random function from a given domain, and to its generalisation to vector processes; (2) mainly based on the vast area of the simulation methods includes directional simulation (DS) and importance sampling (IS), applied either separately or in combination; and (3) represent the well-known statistical approach called response surface method to approximate the limit state function. The advantages and disadvantages of using each category have been discussed elsewhere (e.g., Pinto, [19]).

Later in 90's, the work by Bazzurro and Cornell [3] and Cornell [4] tried to compare the seismic demand and capacity of building as in the basic reliability formulation for the static case. The seismic demand was determined as the maximum response of the structure during the dynamic analysis with the specific level of seismic action. This method is called "SAC-FEMA method" which has the advantage of providing the closed-from solution to compute the probability of failure (P_f). In addition to SAC-FEMA method, the PEER method, which has several conceptual similarities with the first, is not in closed-form but it allows more flexibility and generality in the evaluation of the desired so-called "decision variable", not necessarily coinciding with P_f.

3. Performance Assessment of buildings

The probabilistic seismic performance assessment can be performed using currently available two classes of methods. The first method is more practice-oriented and widely accepted as a standard tool for performance assessment, called "conditional probabilistic approach or an IM-based approach". The second one is more advanced and requires strong knowledge in probability theory and random process, called "unconditional probabilistic approach".

3.1 Conditional probability approach (IM-based methods)

In IM-based methods, one or more ground motion intensity is used as an interface to link the seismology and structural response. Firstly, the structural response (i.e., drift demand) as function of ground motion intensity or intensities is developed and integrated with seismic hazard curve to produce a structure specific drift hazard curve, $H_D(d)$; which provides the annual probability that the drift demand D exceeds any specified value d. Then, the drift hazard curve is jointed with the drift capacity representation to estimate the annual probability of exceedance (λ_{LS}) of a specific of performance level (i.e., the probability of performance level not being met).

Using the total probability theorem (Benjamim and Cornell, [6]), the discrete form of $H_D(d)$ can be estimated as given in Cornell et al. [10]:

$$H_D(d) = P(D \geq d) = \sum_{all\ x_i} P(D \geq d \mid S_a = x_i) P(S_a = x_i) \quad (1)$$

where S_a is a spectral acceleration considered as the *IM*.

The probability of exceedance of drift is expanded by conditioning on all possible levels of the ground motion, as can be seen in Eq.(1). The likelihood of given level of spectral acceleration, $P(S_a = x)$, can be determined form the standard hazard curve $H(S_a)$. The advanced nonlinear dynamic analysis of structure can be used to estimate the $P(D \geq d \mid S_a = x)$, the likelihood that the drift exceeds d given that the value of S_a is known.

The continuous from the Eq. (1) is:

$$H_D(d) = \int P(D \geq d \mid S_a = x) \left| dH(x) \right| \quad (2)$$

where $|dH(x)|$ is the absolute value of the derivative of the site's spectral acceleration hazard curve times dx.

Using the total probability theory again, the annual probability of exceedance (λ_{LS}) of a specific level of performance can be estimated as follows:

$$\lambda_{LS} = P(C \leq D) = \sum_{all\ d_i} P(C \leq D \mid D = d_i) P(D = d_i) \quad (3)$$

The likelihood of a given displacement demand level $P(D=d)$ can be determined from the drift hazard curve derived in Eq. (1) or (2).

The IM-based methods have been the subject of a considerable body of research. Closed-form solutions of Eq. (3) have been proposed [10][23][27]. Various studies have been devoted at checking the approximation made [1][7] and reaching the conclusion that the main source of error is in the linear fit (in log-log space) of the seismic hazard curve. The latter limitation, on the other hand, is relaxed with consideration of second-order logarithm formulation [23][27][16].

While the issue of approximation of closed-form solutions is of great relevance to the adoption of the PBEE paradigm by the practicing engineers, other issues raise greater concerns. The first is the accuracy of IM-based methods in a wider sense. Eq. (3) rests on the assumption that, given S_a, the demand D is independent of all other ground motion properties, which is called the sufficiency property of the IM [23]. While it is obvious that λ_{LS} is theoretically unique, multiple studies have highlighted how estimations obtained by Eq. (3) exhibit a non-negligible dependence upon the chosen IM (Rajeev *et al.*, [22]). In order to ensure high accuracy in the assessment of structural performance via Eq. (3), the suitable IM can be selected using efficiency and sufficiency conditions (Luco [17]; Luco and Cornell [18]). An IM that exhibits these properties will tend to be structure-specific, recognising both the important modes of vibration and effects of nonlinear behaviour as well as the frequency content of the earthquake records. An efficient IM is defined as one that results in relatively small variability of structural responses for a given IM level as measured. Figure (1) shows the comparison of efficiency of two intensity measures with respect to interstory drift angle θ_{max}: on the top, elastic spectral displacement (S_{de}) and on the bottom, inelastic spectral displacement (S_{di}) as shown in Tothong [25]. The dashed vertical line represents the drift level at yielding, as determined from static pushover analysis. The circles indicate where global dynamic instability of the structure is reached. The counted-median and the 16% and 84% fractiles are shown with solid and dashed-dotted lines, respectively. Figure (1) indicates that $\sigma_{\ln IM}$ is reduced by about 50% when using S_{di} instead of S_{de}, implying that the number of records needed to achieve the same accuracy in estimating the mean $\ln\theta_{max}|IM$ can be reduced by a factor of four. The corresponding reductions in $\sigma_{\ln\theta_{max}|IM}$ can also be directly observed in Figure (1), by comparing the distances between the 16th and 84th fractiles of θ_{max} for a given S_{di} with those obtained employing S_{de}.

A sufficient IM is one for which the conditional probability distribution of demand (D) given IM is independent of other ground motion parameters, such as those involved in computing the seismic hazard, i.e., the magnitude M, the distance R, and ε (the number of standard deviations by which an observed logarithmic spectral acceleration differs from the mean logarithmic spectral acceleration of an attenuation equation). A sufficient IM is desirable because it implies that any set of ground motions selected for nonlinear dynamic analysis of the structure will result in approximately the same $P(D>d|IM_1) \approx P(D>d|IM_2)$. Figure (2) shows a comparison of the sufficiency with respect to ε of elastic spectral acceleration at first-mode period of the structure (top) and inelastic spectral displacement (bottom).

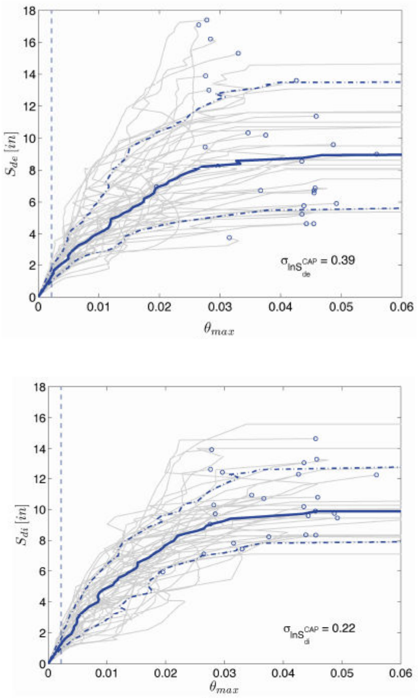

Figure 1: Comparison of the “efficiency” of the IM (adopted from Tothong [25])

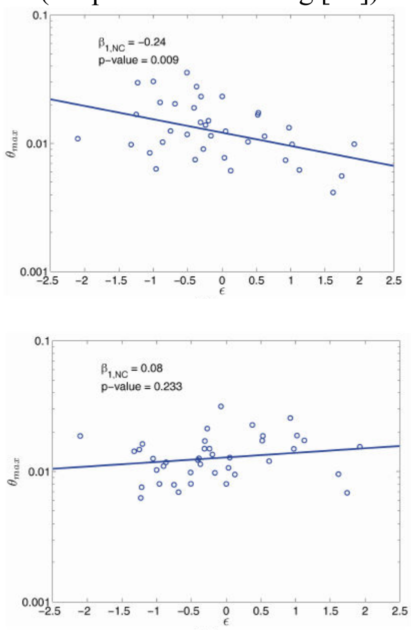

Figure 2: Comparison of the “sufficiency” of the IM (adopted from Tothong [25])

It can be seen from the figure that the response highly depends on ε, when the elastic spectral acceleration at first-mode period is considered as the *IM* (see the slope of the line $\beta_{I,NC}$ and the *p*-value). Conversely, the response negligibly depends on ε, when the inelastic spectral displacement is considered as the *IM* (see again the slope of the line $\beta_{I,NC}$ and the *p*-value). Inelastic spectral displacement is a more effective a scalar *IM* than elastic spectral acceleration at first mode period.

4. Commonly used Intensity Measures (*IM*) in probabilistic structural assessment

Selection of ground motion time-histories is an important consideration when seismic assessment of a structure is based on dynamic analysis. Careful ground motion time-history selection can achieve the same reduction in bias and variance of structural response as is gained by more advanced measures of ground motion intensity, while allowing the user to process the time-histories using simple measures of intensity such as elastic spectral acceleration (Shome and Cornell [24], Baker and Cornell [2]). This said, much efficiency (and sufficiency) can be gained by using an appropriate IM in assessing structural performance.

The peak ground acceleration of a record has been commonly used in the past. More recently, spectral response values have been used as *IM*. For example spectral acceleration at the first mode period T_1 has been shown to be more *efficient* than PGA, mostly because $S_a(T_1)$ is structure-specific. Later, the use of $S_a(T_1)$ has been shown to lead to biased estimates of response for tall and long-period structures and near-source ground motions by Shome and Cornell [24]. This is because for tall, long-period buildings, the higher modes typically contribute significantly to the seismic response (Shome and Cornell [24]). Moreover, for long-period structures $S_a(T_1)$ has been observed to be rather *insufficient* as well (i.e., given $S_a(T_1)$, response still depends on *M*). Like the observed *inefficiency*, this *insufficiency* is again due to the fact that $S_a(T_1)$ cannot reflect higher-mode spectral accelerations, which, conditional on $S_a(T_1)$, are dependent on *M*. Note, in addition, that for soft-soil or near-source ground motions with a predominant period near e.g., the second-mode period of the structure T_2, the intensity measure $S_a(T_1)$ may prove particularly inefficient. This drawback in single-valued *IM*s stimulated researchers to find alternative vector-valued IMs incorporating $S_a(T_1)$ or better scalar-valued *IM* that can more effectively predict the response of a structure.

A number of research studies have been carried out by different people [e.g., Cordova *et al.* [8], Vamvatsikos and Cornell [26], Baker and Cornell [2]] to find better scalar-valued *IM* or vector-valued *IM* that can more effectively capture important features of the ground motion. Using spectral shape ($R_{T1,T} = S_a(T)/S_a(T_1)$), magnitude (*M*), distance (*R*), or epsilon (*ε*), together with $S_a(T_1)$ as second component of a 2-components **IM** vector for assessment of structures has been considered in the past. Shome and Cornell [24] and Bazzuro [5] have considered a vector **IM** comprised of $S_a(T_1)$ and the ratio $S_a(T_2)/S_a(T_1)$, as well as a scalar *IM* that combines $S_a(T_1)$ and $S_a(T_2)$. The study of Cordova *et al* [8] introduced an improved intensity measure that takes into account the inelastic lengthening of the period. Luco and Cornell [18] studied several scalar-valued *IM*'s which can effectively capture the response of structures subjected to both near-source and ordinary earthquakes. The recent study by Tothong [25] explores ground motion *IM*s such as inelastic spectral displacement S_{di}, and S_{di} corrected by a higher-mode factor. The more details on the selection of *IM* and its efficiency and sufficiency can be found in Rajeev [20].

4. Prediction of Structural fragility

The structural fragility [i.e., $P(D>d|S_a=x_i)$] is equal to the probability that the performance measure *D* is larger than specified demand level as a function of the intensity measure level. This can be calculated using either numerical integration or a closed-form solution.

By making the assumption that the distribution of the demand for a given level of the *IM* is described by a lognormal distribution, the fragility function can be estimated as follows:

$$P(D > d|S_a) = 1 - \Phi\left(\frac{\ln d - \ln \eta_{D|IM}}{\beta_{D|IM}}\right) \qquad (4)$$

In the applications, however, a problem often arises with the evaluation of the parameters $\eta_{D|IM}$ and $\beta_{D|IM}$. Since structural response is evaluated by means of nonlinear time-history analyses, it is not uncommon to encounter numerical instabilities which erroneously affect the parameters estimation. In these cases, the fragility function can be calculated as follows:

$$P(D > d \mid IM) = P(D > D \mid IM, \bar{c}) \cdot (1 - P(c \mid IM)) + P(c \mid IM) \tag{5}$$

where c and $\bar{c}$ denote the collapse and non-collapse situations respectively, $P(c|IM)$ is the probability of having a collapse (identified as very large D values) for a given IM and $P(Y>1|IM, \bar{c})$ is the fragility given that no collapse has occurred, which can be again assumed to be described by a Lognormal distribution:

$$P(D > d \mid IM, \bar{c}) = 1 - \Phi\left(\frac{\ln d - \ln \eta_{D|IM,\bar{c}}}{\beta_{D|IM,\bar{c}}}\right) \tag{6}$$

where $\eta_{D|IM,\bar{c}}$ and $\beta_{D|IM,\bar{c}}$ are the median and logarithmic standard deviation of D given IM and $\bar{c}$. Use of Eq. (5) allows accounting separately for "converged" and "non-converged" values. It is important to stress, however, that the validity of Eq. (5) rests on the assumption that the "failure set" includes the "numerical non-convergence set", i.e., that all numerical non-convergence cases can be considered as failures.

The closed-form solution for structural fragility and then the mean annual frequency of limit state exceedance can be derived by making following assumptions. First, assume that the site hazard curve can be approximated as linear on a log-log plot in the region of interest

$$H(S_a) = k_0 \cdot S_a^{-k} \tag{7}$$

Typical values of the important log-log slope k are 1 to 4. The demand on the structure for a given spectral acceleration can be interpreted as:

$$D = \eta_{D|S_a} \cdot \varepsilon_{D|S_a} \tag{8}$$

where it is assumed that the median is a power-law function of spectral acceleration level ($\eta_{D|S_a} = a \cdot S_a^b$) and that $\varepsilon_{D|S_a}$ is a unit-median Lognormal variable with dispersion equal to $\beta_{D|S_a}$.

The closed form solution can be expressed as:

$$\lambda_{LS} = H(S_a)\left(1/\sqrt[b]{a}\right) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{Y|S_a}} \tag{9}$$

where $1/\sqrt[b]{a}$ is the spectral acceleration value that correspond to a median D value (Jalayer [14]).

The nonlinear dynamic analyses can be used to build the relationship between the demand and spectral acceleration as in Eq. (8). One procedure, also known as the "Cloud Analysis" (Jalayer *et al.* [15]), is a convenient choice (though not the most accurate). An advantage of this method is that it is based on the ground motions as they are recorded and does not require scaling. The procedure consists of applying a suite of ground motion records (in the order of 10-30 records) to the structure and to calculate the demand D. Then, by performing a simple linear regression of the logarithm of D against the logarithm of S_a, one can obtain the parameters a and b. Figure 3 shows the result Cloud Analysis and the power-law fit for the demand. However, the accuracy in the predicting the median demand depends on the selection of the ground motion that should cover the structural response from the linear to nonlinear behaviour. This may not be guaranteed in all the situations.

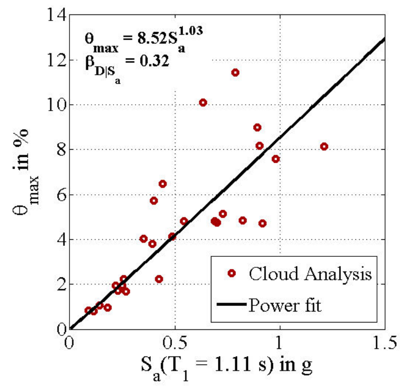

Figure 3: The power-law relationship between the interstory drift and spectral acceleration (adopted from Rajeev and Tesfamariam [21]

Another approach is to use the incremental dynamic analysis (IDA) which requires scaling of selected records at different levels of S_a's. Therefore, unlikely in the Cloud Analysis, the relationship for $\beta_{D|S_a}$ with S_a can be developed. However, the computational time is extensively high in comparison to Cloud Analysis.

In order to overcome the limitations in both Cloud Analysis and IDA, Rajeev et al., [22] proposed an alternative procedure to select record set that can cover the structural response from liner to nonlinear range and an efficient method to estimate the collapse fragilities. The approximate procedure is outlined here:

1. Estimate fractile IDA curves of the structure by means of pushover analysis, using a tool such as ‘SPO2IDA’ [28]. This requires a piece-wise linear fit of the pushover curve (Fig. 4, top).
2. Use these approximate IDA (Fig. 4, bottom) to get an estimate of the upper bound collapse intensity (s_C). Select records to span in an approximately uniform manner the intensity range [s=0, s=s_C]. Records should also be selected at least with reference to the causative events (magnitude and distance bins) from PSHA of each sub-interval in which [s=0, s=s_C] is divided (these need not to be large in number).
3. Perform cloud analysis and collect intensity-response data points for all responses of interest, as shown in Fig. 5.
4. Identify outliers and carry out regression analysis to fit the median model in Eq. (8) with a *constant* dispersion to non-collapse points.
5. Use the approximate IDA from step 1 to evaluate median and dispersion of the collapse intensity, $s_{C,50\%}$ and β_{sc} parameters of the approximate lognormal collapse fragility.

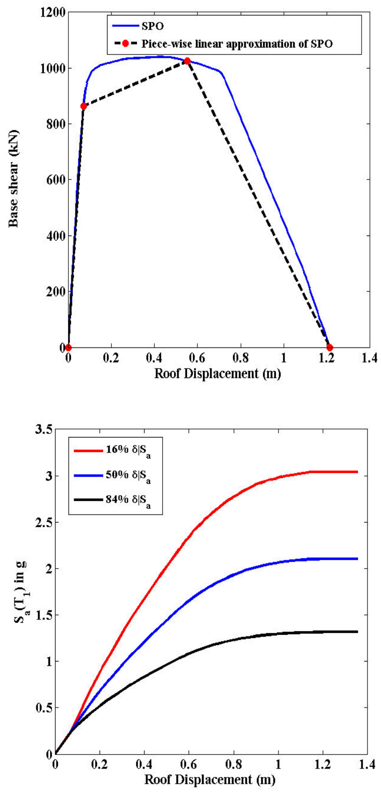

Figure 4: Pushover curves, piece-wise linear approximations and approximate IDA from SPO2IDA (adopted from Rajeev et al., 2014)

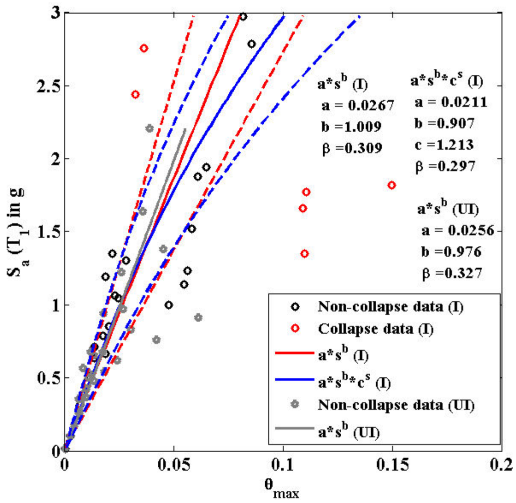

Figure 5: Cloud analysis and the median non-collapse demand model fit (solid line) together with the 16th and 84th percentile (dash line) [adopted from Rajeev et al., [22]]

The method is computationally cheaper than IDA and employs dynamic analysis as opposed to other approximate methods that rely on nonlinear static one, thus accounting for record-to-record variability and cyclic degradation, which are very important especially for non code-conforming structures.

5. Conclusions

This paper provides review on the probabilistic performance based earthquake engineering and its recent advancements. This also outlines the basic equations used in design and assessment of structures. Special consideration is given to the commonly accepted IM-based approach. Further, the methods to compute the dynamic response of structure with in the performance based framework.

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