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# FINITE DIFFERENCE APPROXIMATION TO WAVE EQUATION TO SIMULATE SLOSHING OF GROUND SUPPORTED TANKS FOR EARTHQUAKE LOADINGS

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**Abstract:** Sloshing is a liquid vibration physical phenomenon which causes when liquid storage tank is subjected to external loading. Major effects due to sloshing are higher impact pressure on tank walls and over spillage of liquid. Therefore, this study was aimed to investigate the sloshing pressure of ground supported rigid cylindrical tanks under earthquake loading. In this study, wave equation was used to convert the physical phenomenon to a mathematical model and nonlinear terms were approximated. Finite difference method was used to solve the mathematical model for the simulation of sloshing in frequency domain for 2D analysis. Input motions of earthquake loading were obtained from the average Fourier spectrum of seven earthquake records. Here, the liquid was assumed to be inviscid, incompressible and irrotational. Based on the results obtained using the generated finite difference code, the aspect ratio of the tank and frequency of ground motion affects the sloshing pressure.

**Keywords:** earthquake; finite difference; frequency domain; ground supported tanks; sloshing;

## 1. Introduction

Sloshing is a physical phenomenon which is caused when liquid tanks are subjected to movement or vibration by external loading. Higher impact pressure on tank walls and spill over of liquid are major effects of sloshing. Higher impact pressure will cause instability to the structure which leads to structural failures whereas tanks with acids, fuel will cause severe environmental problems due to over spilling. These effects should be analysed to avoid damages.

Analytical, numerical and experimental techniques were used in previous studies. Mass-spring model (Malhotra, P., 1997), potential flow theory (Zhang, H. and Sun, B., 2014), Laplace and Bernoulli equations (Ruiz, R.O. et al, 2015) are some of the models used to study sloshing in past researches.

The present work focuses on developing a general finite difference code for the simulation of sloshing pressure of ground supported tanks for earthquake loadings. In this study, wave equation in frequency domain was used to develop a mathematical model and finite difference method was applied to solve the mathematical model.

## 2. Formulation of Mathematical Model for Sloshing

A schematic diagram of a 2D cylindrical tank and the coordinate system as shown in Figure 1 was considered throughout the study. The tank was assumed to be rigid, fixed at the bottom and top as free surface. The liquid was assumed as inviscid, incompressible and irrotational.

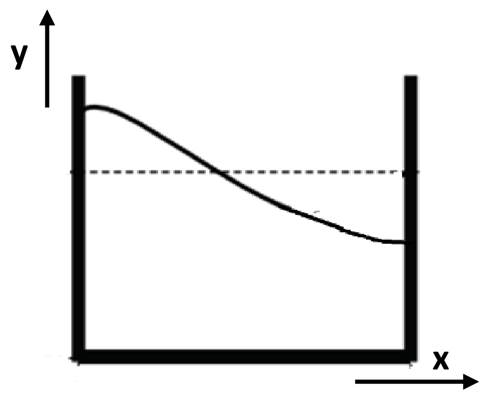


Fig 1: Schematic diagram of 2D cylindrical tank

Wave equation for pressure was considered,

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} \qquad (1)$$

Here, *t*, *P*, *c* are time, pressure and P-wave velocity in the water respectively.

But, pressure can be defined as a simple harmonic motion for sloshing.

$$P = P_0 e^{i\omega t} \qquad (2)$$

From equation (1) and (2), wave equation in frequency domain can be derived as in equation (3)

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = -\frac{\omega^2}{c^2} P \qquad (3)$$

Since sloshing occurs due to external loading, a source term *B* was introduced to equation (3) which implies equation (4). Here *B* is external loading as pressure function.

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\omega^2}{c^2} P = -B \qquad (4)$$

Equation (4) was converted as finite difference equation by applying central finite difference method as follows:

$$\frac{P_{x+\Delta x,y} - 2P_{x,y} + P_{x-\Delta x,y}}{(\Delta x)^2} + \frac{P_{x,y+\Delta y} - 2P_{x,y} + P_{x,y-\Delta y}}{(\Delta y)^2} \qquad (5)$$

$$+\frac{\omega^2}{c^2} P_{x,y} = -B_{x,y}$$

Equation (5) represents the sloshing as mathematical model. Here, ω is the frequency of external loading.

**3. Solution for the Mathematical Model**

To solve the finite difference equation accurately and easily, a finite difference code was written using Matlab based on the following procedure.

2D rectilinear mesh with the size of nx × ny and having constant grid spacing of b and h along the x and y directions accordingly was considered. The terms were grouped by grid location as shown in Figure 2 to solve the equation (6) implicitly.

Where the subscripts i and j were used as local references to discrete locations on the grid.

$$\frac{\omega^2}{c^2} P_{i,j} + \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{b^2}$$

$$+\frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{h^2} = -B_{i,j} \qquad (6)$$

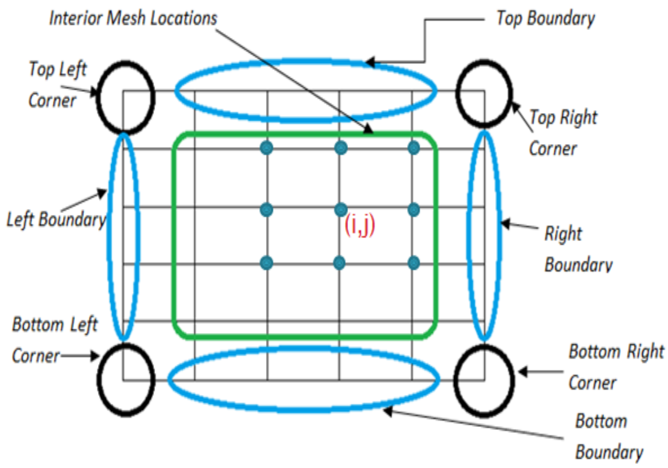


Fig 2: Grid locations

The equation (6) was solved for the coefficient set as shown below which operates on $P_{i,j}$. The coefficient sets represents the amount of contribution to pressure coming from the adjacent nodes to the considered point as in Figure 3. $TLC_{i,j}$, $TRC_{i,j}$, $BLC_{i,j}$, $BRC_{i,j}$, $T_{i,j}$, $B_{i,j}$, $R_{i,j}$, $L_{i,j}$, $M_{i,j}$ represents the contributions from top left corner, top right corner, bottom left corner, bottom right corner, top, bottom, right, left and middle sides respectively.

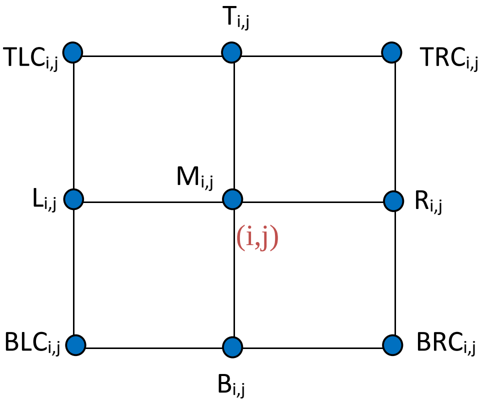


Fig 3: Symbolic notations for the different cardinal directions.

Equation (6) was rearranged as follows and coefficient sets for the internal nodes were obtained.

$$\underbrace{\left\{-\frac{\omega^2}{c^2}(bh)+2\left(\frac{h}{b}+\frac{b}{h}\right)\right\}}_{M_{i,j}}P_{i,j}\underbrace{-\frac{h}{b}}_{R_{i,j}}P_{i+1,j}$$

$$\underbrace{-\frac{h}{b}}_{L_{i,j}}P_{i-1,j}\underbrace{-\frac{b}{h}}_{T_{i,j}}P_{i,j+1}\underbrace{-\frac{b}{h}}_{B_{i,j}}P_{i,j-1}=B_{i,j}(bh) \qquad (7)$$

It can be noted that there is no contribution coming from adjacent corner nodes to the considered node i,j.

Coefficient sets for the top, bottom, right side and left side boundaries and for the corners also were derived as follows.

Coefficient set for the top boundary,

$$M_{i,j}=\left\{-\frac{\omega^2}{c^2}\left(\frac{bh}{2}\right)+\left(\frac{h}{b}+\frac{b}{h}\right)-\frac{\omega^2}{g}b\right\}$$

$$R_{i,j}=-\frac{h}{2b}$$

$$L_{i,j}=-\frac{h}{2b}$$

$$T_{i,j}=0$$

$$B_{i,j}=-\frac{b}{h}$$

Coefficient set for the bottom boundary,

$$M_{i,j}=\left\{-\frac{\omega^2}{c^2}\left(\frac{bh}{2}\right)+\left(\frac{h}{b}+\frac{b}{h}\right)\right\}$$

$$R_{i,j}=-\frac{h}{2b}$$

$$L_{i,j}=-\frac{h}{2b}$$

$$T_{i,j}=-\frac{b}{h}$$

$$B_{i,j}=0$$

Coefficient set for the right boundary,

$$M_{i,j}=\left\{-\frac{\omega^2}{c^2}\left(\frac{bh}{2}\right)+\left(\frac{h}{b}+\frac{b}{h}\right)\right\}$$

$$R_{i,j}=0$$

$$L_{i,j}=-\frac{h}{b}$$

$$T_{i,j}=-\frac{b}{2h}$$

$$B_{i,j}=-\frac{b}{2h}$$

Coefficient set for the left boundary,

$$M_{i,j}=\left\{-\frac{\omega^2}{c^2}\left(\frac{bh}{2}\right)+\left(\frac{h}{b}+\frac{b}{h}\right)\right\}$$

$$R_{i,j}=-\frac{h}{b}$$

$$L_{i,j}=0$$

$$T_{i,j}=-\frac{b}{2h}$$

$$B_{i,j}=-\frac{b}{2h}$$

Coefficient set for the top right corner boundary,

$$M_{i,j} = \left\{ -\frac{\omega^2}{c^2}\left(\frac{bh}{4}\right) + \frac{1}{2}\left(\frac{h}{b} + \frac{b}{h}\right) - \frac{\omega^2}{g}\left(\frac{b}{2}\right) \right\}$$

$$R_{i,j} = 0$$

$$L_{i,j} = -\frac{h}{2b}$$

$$T_{i,j} = 0$$

$$B_{i,j} = -\frac{b}{2h}$$

Coefficient set for the bottom right corner boundary,

$$M_{i,j} = \left\{ -\frac{\omega^2}{c^2}\left(\frac{bh}{4}\right) + \frac{1}{2}\left(\frac{h}{b} + \frac{b}{h}\right) \right\}$$

$$R_{i,j} = 0$$

$$L_{i,j} = -\frac{h}{2b}$$

$$T_{i,j} = -\frac{b}{2h}$$

$$B_{i,j} = 0$$

Coefficient set for the top left corner boundary,

$$M_{i,j} = \left\{ -\frac{\omega^2}{c^2}\left(\frac{bh}{4}\right) + \frac{1}{2}\left(\frac{h}{b} + \frac{b}{h}\right) - \frac{\omega^2}{g}\left(\frac{b}{2}\right) \right\}$$

$$R_{i,j} = -\frac{h}{2b}$$

$$L_{i,j} = 0$$

$$T_{i,j} = 0$$

$$B_{i,j} = -\frac{b}{2h}$$

Coefficient set for the bottom left corner boundary,

$$M_{i,j} = \left\{ -\frac{\omega^2}{c^2}\left(\frac{bh}{4}\right) + \frac{1}{2}\left(\frac{h}{b} + \frac{b}{h}\right) \right\}$$

$$R_{i,j} = -\frac{h}{2b}$$

$$L_{i,j} = 0$$

$$T_{i,j} = -\frac{b}{2h}$$

$$B_{i,j} = 0$$

Equation (7) can be further modified into matrix form as follows:

$$[K][P] = [B]$$

$$[P] = [K]^{-1}[B] \quad (8)$$

$[K]$ can be obtained from the aforementioned coefficient sets.

$[B]$ can be found by substituting external loading as pressure. In this study, the bottom of the tank was considered as fixed and the top surface was free. Thus, external loading was assumed to be transferred to the tank from the two rigid side boundaries.

Using the equation (8) sloshing pressure at each node can be found. A Matlab code was generated to do the simplifications quickly and easily for the aforementioned simulation procedure.

The input parameters **w** (frequency) and **a** (amplitude) can be obtained from Fourier spectrum of earthquake loading.

**4. Analysis for the Possible Earthquake Loading of Sri Lanka**

Analysis was performed for the selected case studies in Table 1 by applying possible level of earthquake loading in Sri Lanka.

### 4.1 Case Studies

In order to find out the effect of aspect ratio in sloshing pressure four case studies were considered.

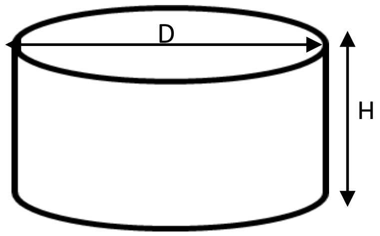


Fig 4: Details of tank

Table 1: Case studies

| Aspect Ratio (D/H) | Diameter (m) | Height (m) |
|---|---|---|
| 0.5 | 1.5 | 3 |
| 2 | 6 | 3 |
| 4 | 10 | 2.5 |
| 6 | 12 | 2 |

### 4.2 Selection of Earthquake Loading

According to Euro code 8, minimum of seven earthquake records are needed to find the average spectrum for analysis.

Earthquake records were obtained from PEERC (Pacific Earthquake Engineering Research Centre) network as accelerograms and response spectra of those records were obtained. Seven accelerograms were selected in such a way that the average response spectrum of the selected earthquake records matched with the available design response spectrum of Sri Lanka (Uduweriya. et al, 2013) as shown in Figure 5.

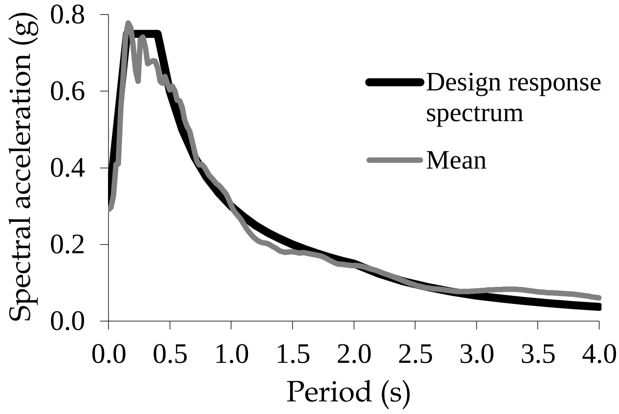


Fig 5: Average response spectrum

### 4.3 Applying Earthquake Loadings to the Code

Collected earthquake data were accelerograms. It is required to find out the frequency and amplitude of those data since those are important input parameters in the generated finite difference code for the simulation.

Since the earthquake data contain discrete values, to find the frequency and amplitude, Fast Fourier transform algorithm was used. Matlab code was generated and used for this computation and those values were verified from the results obtained using the software SeismoSignal.

Figure 6 shows the average Fourier spectrum which was used for the analysis.

Using the amplitude values of each frequency from average Fourier spectrum, earthquake loading was calculated as pressure value. These pressure values were substituted to the external load matrix (*B*) in the code.

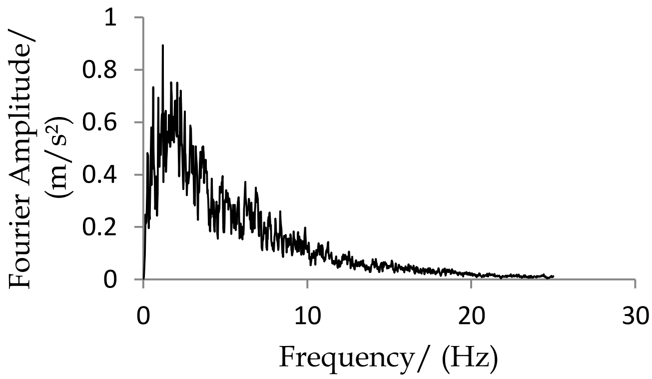


Fig 6: Average Fourier spectrum

Since, the base was assumed to be fixed and top surface to be free, external load was considered as transferring from the two rigid side walls. Since the walls were treated as rigid the same amount of external loadings applicable for those boundaries.

Input values for frequency and amplitude for matlab code were obtained from Figure 6.

## 4. Results and Discussion

The generated code will give the sloshing at each node of meshed tank as output. This code is needed to be run required number of times by changing the size of mesh in order to get accurate values. Required mesh size to get converged accurate results was varied with the size of tanks.

Figure 7 shows the variation of sloshing pressure with mesh size for case study 2 (D=6m and H=3m)

Results obtained for the case study 2 at the peak point of average Fourier spectrum (1.196 Hz and 0.894 m/s²) is shown in Figure 8. Size of the mesh to get the converged answer for the case study 1 was 0.025m and for case studies 2,3 and 4 was 0.05m.

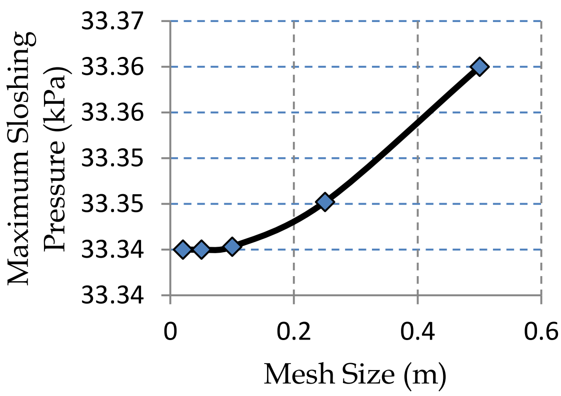


Fig 7: Variation of sloshing pressure with mesh size

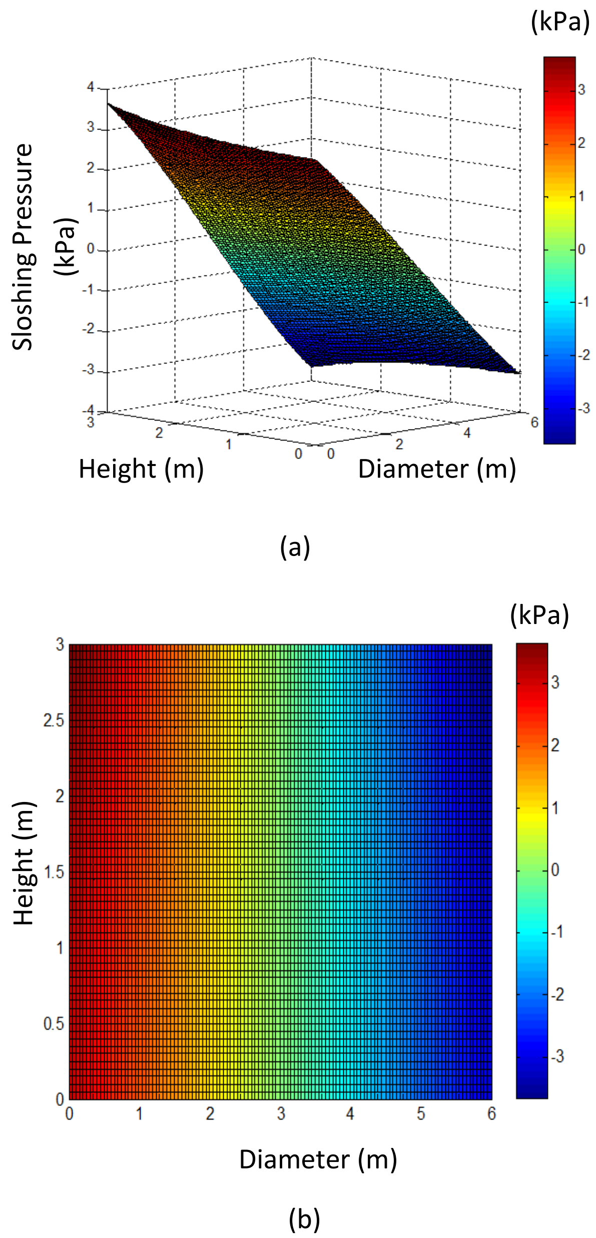


Fig 8: Variation of sloshing pressure across the tank (H=3, D=6m)

From Figure 8, higher effect due to sloshing pressure occurs at the side walls of the tank whereas at the middle sloshing pressure is zero.

Figure 9 shows the variation of normalized maximum sloshing pressure (normalized with respect to static pressure of each tank) with the frequency of input motion from the above analysis.

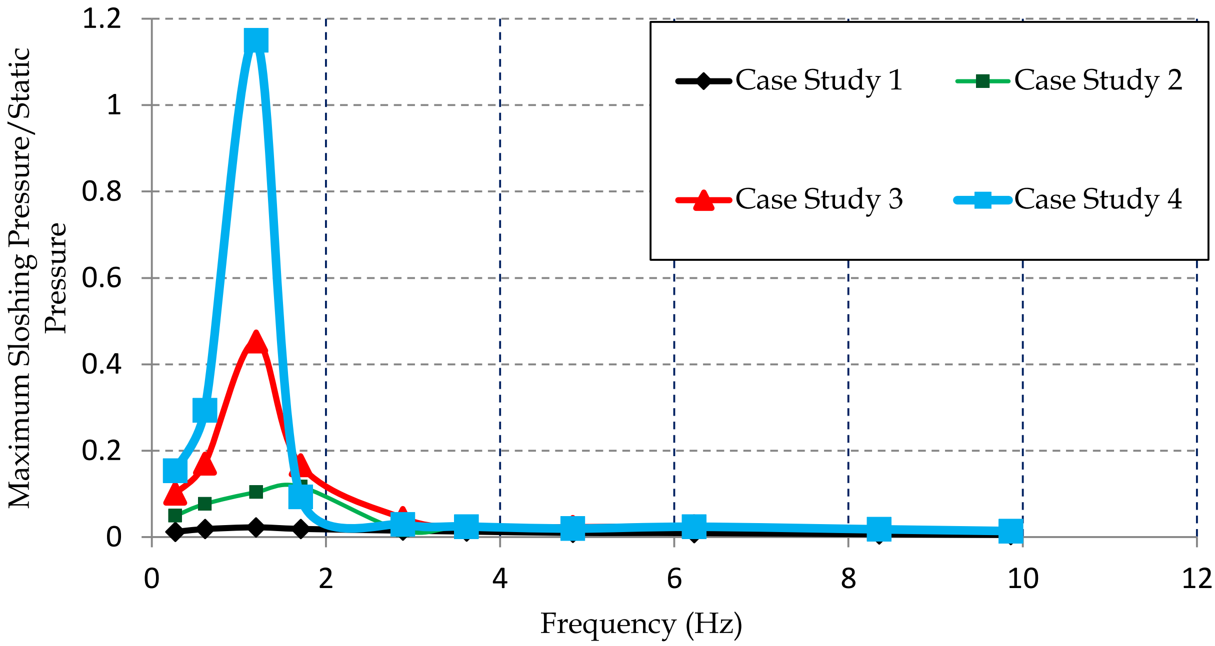


Fig 9: Variation of normalized sloshing pressure with frequency

According to Figure 9, sloshing pressure effect is higher for the frequency range of 0.1Hz – 3Hz. When frequency increases, waves interfere destructively. Thus beyond 3Hz sloshing effect is inappreciable.

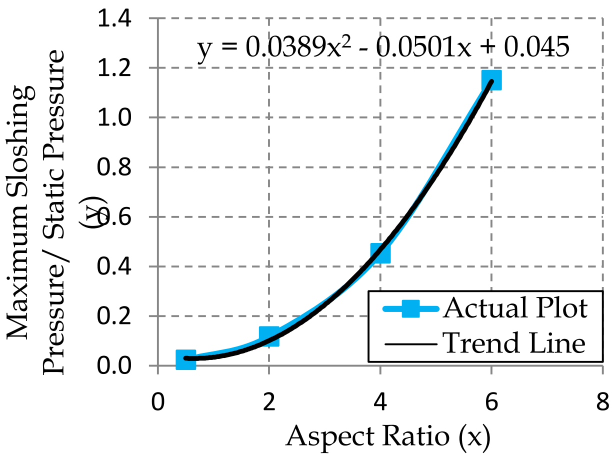


Fig 10: Variation of normalized sloshing pressure with aspect ratio

Figure 10 shows the variation of normalized sloshing pressure with aspect ratio for the possible earthquake shaking of Sri Lanka. The variation shows a second order polynomial trend.

## 5. Conclusions and Recommendations

Generated finite difference code can be used to simulate sloshing pressure of ground supported rigid cylindrical tanks with fixed bottom and free top surface.

From the results obtained,

Sloshing pressure is dominant for the frequency range of 0.1 Hz – 3 Hz irrespective of aspect ratio. Beyond this limit the effect of sloshing is insignificant.

Sloshing pressure increases with aspect ratio. But for the aspect ratios less than 1 sloshing effect is inappreciable.

Therefore, sloshing effect for the tanks with aspect ratio more than 1 should be necessarily checked for the earthquake loading within the frequency range of 0.1 Hz – 3 Hz to avoid the ground supported water tank failures due to sloshing in Sri Lanka.

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