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MODIFIED HYPERBOLIC SHEAR DEFORMATION THEORY FOR STATIC FLEXURE ANALYSIS OF THICK ISOTROPIC BEAM

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Abstract: A hyperbolic shear deformation theory for thick isotropic beams is developed where the displacements are defined using a meaningful function which is more physical and directly comparable with other higher order theories. Governing variationally consistent equilibrium equations and boundary conditions are derived in terms of the stress resultants and displacements using the principle of virtual work. This theory satisfies shear stress free boundary condition at top and bottom of the beam and doesn't need shear correction factor. Results obtained for stresses and displacements using the present theory for static flexure of simply supported uniform isotropic beam carrying uniformly distributed load are compared with other beam theories and the exact elasticity solution.

Keywords: shear deformation theory, static flexure, stress resultants, thick isotropic beam.

1. Introduction

Several beam theories are used to represent the kinematics of deformation. Among those Euler beam theory (EBT) is earliest and one of the well - known theory which has a major drawback of neglecting effects of transverse shear strain because of the assumption the plane section that is perpendicular to neutral axis of beam before bending, remains plane and perpendicular to axis after the deformation. This theory provides excellent solution for the analysis of slender beams whereas for moderately short or thick beams, the solutions are not in the acceptable range.

2. Literature Review

In the development of beam theories, Timoshenko [15] was the first to include the influence of transverse shear strain and rotatory inertia effect into the newly developed first order shear deformation theory (FSDT). In the Timoshenko beam theory, it is assumed that cross section remains plane but not normal to the neutral axis after deformation. Since Timoshenko beam theory assumes a constant transverse shear stress distribution through the beam depth, it is necessary to have shear correction factor for the beam. Cowper [2,3] analysed the accuracy of Timoshenko beam theory for transverse vibration of simply supported beam with respect to fundamental frequency and reported some values for shear correction factor of beams having various cross section.

The limitations on the Euler beam theory and the Timoshenko theory have led to the development of higher order theory. Many higher order theories are available in literatures for static and dynamic analysis of the beams. Levinson [7] developed new rectangular beam theory for static and dynamic analysis of the beam where he derived governing equations for beam using vector mechanics. Bickford [1] used the same displacement function used by the Levinson and derived a variationally consistent shear deformation theory for isotropic beams. Third order plate theory developed by Reddy [9] was specialized into beam theory (HSBT) by Heyliger and Reddy [6] to study the linear and non-linear bending and vibration of isotropic beams. These parabolic shear deformation theories obviate the need for the shear correction factor since shear stress free boundary condition in top and bottom of the beam are satisfied.

There is another set of refined shear deformation theories using trigonometric and hyperbolic profiles to define the displacement function. Touratier [17] presented a trigonometric shear deformation theory. However, this theory does not satisfy shear stress free boundary condition. Ghugal and Shimpi [5] developed variationaly consistent trigonometric shear deformation theory (TSDBT) which satisfies the shear stress free function condition at top and bottom surfaces of the beam. Soldatos [14] developed hyperbolic shear deformation theory for homogeneous monoclinic plates. Ghugal and Sharma [4] and Sayyad and Ghugal [12] developed a variationally consistent refined hyperbolic shear deformation theory (HPSBT) for flexure and free vibration of thick isotropic beam. Although this theory satisfies shear free conditions at top and bottom of the beam and doesn't need shear correction factor, there is an inconsistency in the relationship for displacement function hence strains, compare to other higher order theories which have been used in unified higher order theory by Simsek and Reddy [13]. Recently, Pankade, Tupe and Salve [8] have developed a hyperbolic shear deformation theory with the displacement function defined using third order variable and hyperbolic function to analyse the isotropic beam.

In the present study, the displacement function used in the hyperbolic shear deformation theory [4] is modified such that the functions used to define the displacements are more physical and directly comparable to other higher order theories. Governing variationally consistent equilibrium equations for uniform isotropic beam are derived in terms of stress resultants and associated force and kinematic boundary conditions are defined in terms of stress resultants and displacements respectively. Solutions for the bending problem of uniform isotropic rectangular beam are derived and associated constants are defined and solutions are validated using an illustrative problem.

In the originally developed hyperbolic shear deformation theory by Ghugal and Sharma [4], the displacement field is assumed as

$$u(x,z) = -z\, dw/dx + [\, z \cosh(1/2) - h \sinh(z/h)]\theta(x)$$

$$w(x,z) = w(x)$$

At the later work of Sayyad and Ghugal, [12] the displacement u(x,z) is given in following form

$$u(x,z) = -z\, dw/dx + [\, z \cosh(1/2) - h \sinh(z/h)][\, dw/dx + \phi(x)]$$

Here u and w are the axial and transverse displacements of the beam center line in the x and z directions, respectively. $\theta(x)$ and $\phi(x)$ are two unknown functions which represent shear rotation and total rotation of cross section at neutral axis respectively. But these functions $\theta(x)$ and $\phi(x)$ are not equal to shear rotation and total rotation of cross section at neutral axis respectively.

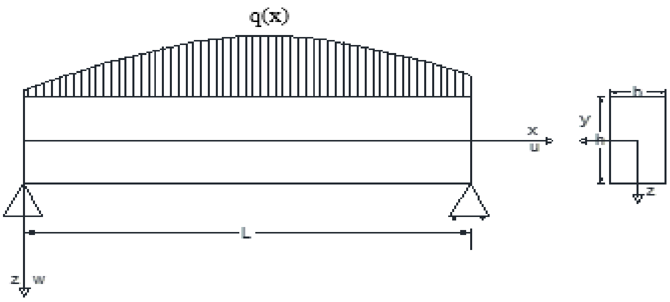
3.Theoretical Formulation of Proposed Modified Beam Theory

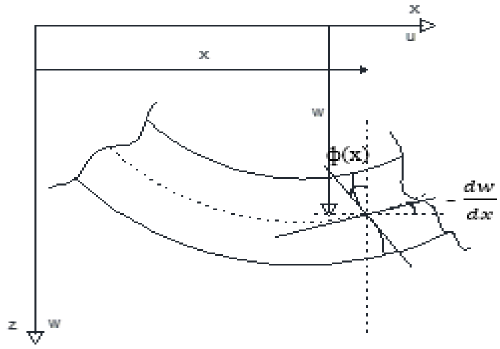
Consider a uniform isotropic thick beam as shown in Fig.1, in which the deformed beam cross section neither stays normal to the deformed centroidal axis nor remains a plane. By using the Cartesian coordinate system (x; y; z) indicated in Fig.1 where the x-axis is coincident with the centroidal axis of the undeformed beam, the y-axis is the neutral axis, and the z-axis is along the thickness of the beam. The beam is subjected to transverse load of intensity q(x) per unit length of the beam.

3.1 Assumptions Made in the Theoretical Formulation

- The in-plane displacement u in x direction consists of two parts:

a. Displacement due to the bending rotation





b. Displacement due to shear rotation which is assumed to be hyperbolic in nature with respect thickness coordinate

Fig 1: Beam under consideration

- The transverse displacement w in z direction is assumed to be a function of x
- One-dimensional constitutive law is used.
- The beam is subjected to lateral load only

3.2 The Displacement Field

Based on the above mentioned assumptions displacement field of the present theory is given as

$u(x,z)= z\,\phi(x)$ -

$\mu(\,h \sinh(z/h)- z)[\,dw/dx + \phi(x)]$ (1)

; $\mu= 1/(\cosh(1/2)\ -1)$

$w(x,z) = w(x)$ (2)

where u(x,z) is axial displacement at any point on the line parallel to beam centroidal axis and also w(x) and ϕ(x) are two unknown functions named the transverse displacement and total rotation of the cross section at neutral axis respectively.

$\theta(x)= [\ dw/dx + \phi(x)]$; ϕ(x) is rotation of cross section due to shear at neutral axis.

The normal strain and transverse strain are obtained using linear theory of elasticity.

$\varepsilon xx = \partial u/\partial x = z\ (d\phi(x))/dx-$

$\mu(\,h \sinh(z/h)-z)[\,dw/dx + \phi(x)]$ (3)

$\gamma xz = \partial u/\partial z + \partial w/\partial x$

$= [\,1 - \mu(\cosh(z/h)-1)][\,dw/dx + \phi(x)]$ (4)

One-dimensional law is used to obtained normal bending and transverse shear stresses.

$\sigma_{xx} = E_{xx}\, \varepsilon_{xx}$ (5)

$\tau_{xz} = G_{xz}\, \gamma_{xz}$ (6)

3.3 Governing Equations and Boundary Conditions

Using above stress and strain relations in Eqns (3)-(6), virtual strain energy δU becomes

$$\delta U = \int_0^L \int_A [\sigma_{zz}\, \delta\epsilon_{xx} + \tau_{xz}\delta\,\gamma_{xz}]\, dA\, dx \qquad (7)$$

and the virtual potential energy of the transverse load q is given by

$$\delta V = -\int_0^L q(x)\delta w\, dx \qquad (8)$$

Applying the theorem of minimum potential energy δπ = δU + δV = 0, it becomes

$$\int_0^L \int_A [\sigma_{xx}\, \delta\epsilon_{xx} + \tau_{xz}\delta\,\gamma_{xz}]\, dA\, dx\ -$$

$$\int_0^L q(x)\delta w\, dx = 0 \qquad (9)$$

By substituting stress resultants and applying integration by parts, we obtain the coupled Euler–Lagrange equations which are the governing differential equations of equilibrium and associated boundary conditions of the beam.

Equation of Equilibrium

$$\frac{d^2 M_{xx}}{dx^2} = -q(x) \qquad (10)$$

$$\frac{d(M_{xx} - M'_{xx})}{dx} - (Q_x - R_x) = 0 \tag{11}$$

Boundary condition

$$\frac{dM_{xx}}{dx} = 0 \quad \text{or} \quad w \text{ is prescribed} \tag{12}$$

$$\bar{M}_{xx} = 0 \quad \text{or} \quad \phi \text{ is prescribed} \tag{13}$$

$$M'_{xx} = 0 \quad \text{or} \quad -\frac{dw}{dx} \text{ is prescribed} \tag{14}$$

Here the stress resultants are defined as follows

$$M_{xx} = \int_A z\, \sigma_{xx}\, dA \tag{15}$$

$$Q_x = \int_A \tau_{xz}\, dA \tag{16}$$

$$M'_{xx} = \mu \int_A [h \sinh(\tfrac{z}{h}) - z]\sigma_{xx}\, dA \tag{17}$$

$$R_x = \mu \int_A [\cosh(\tfrac{z}{h}) - 1]\tau_{xz}\, dA \tag{18}$$

$$V_x = \frac{dM_{xx}}{dx} = Q_x - R_x + \frac{dM'_{xx}}{dx} \tag{19}$$

$$\bar{M}_{xx} = M_{xx} - M'_{xx} \tag{20}$$

Where M_{xx} and Q_x are the usual bending moment and shear force and M'_{xx} and R_x are the higher order stress resultant. V_x is the effective shear force.

For a uniform rectangular isotropic beam, the equations of equilibrium can be obtained in terms of the displacements w and ϕ using the stress resultant-displacement relations given in Eqns (18)-(21).

$$EI\frac{d^4w}{dx^4} - A_0 EI(\frac{d^4w}{dx^3} + \frac{d^3\phi}{dx^3}) = q(x) \tag{21}$$

$$EIA_0\frac{d^3w}{dx^3} - EIB_0\left(\frac{d^3w}{dx^2} + \frac{d^2\phi}{dx^2}\right) + B_0GA(\frac{dw}{dx} + \phi) = 0 \tag{22}$$

Where

$A_0 = \mu\{ \cosh\frac{1}{2} - 12[\cosh\frac{1}{2} - 2\sinh\frac{1}{2}]\}$

$B_0 = \mu^2\{ (\cosh(\frac{1}{2})^2 - 24 \cosh\frac{1}{2} [\cosh\frac{1}{2} - 2\sinh\frac{1}{2}]$

$+6 [\sinh 1 -1]\}$

$C_0 = \mu^2\{\cosh\frac{1}{2} [\cosh\frac{1}{2} - 2\sinh\frac{1}{2}] - \frac{1}{2}[\sinh(1) -1] \}$

3.5 The General Solutions for Static Flexure of Beams

By solving the Eqns (21) and (22), we can obtain the general solutions for w and ϕ

$$\theta = C_2\cosh\lambda x + C_3\sinh\lambda x + \frac{V_x}{EIm} \tag{23}$$

$$EIw = \int\int\int\int q(x)\,dxdxdxdx + EI\frac{A_0}{\lambda}[C_2\sinh\lambda x + C_3\cosh\lambda x] + C_1\frac{x^3}{6} + C_4\frac{x^2}{2} + C_5x + C_6 \tag{24}$$

$$\phi = (1 - A_0)[C_2\sinh\lambda x + C_3\cosh\lambda x] - \frac{1}{EI}(C_1\frac{x^2}{2} + C_4x + C_5 - \frac{V_x}{EIm} + \frac{1}{EI}\int\int\int q(x)\,dxdxdx) \tag{25}$$

Where

$k = \frac{GAC_0}{EIA_0}$ $m = B_0/A_0 - A_0$ and $\lambda = k/m$

4. Illustrative Example

A simply supported beam with rectangular cross section (b× h) is subjected to uniformly distributed load q over the span L at surface

$z = -h/2$ acting in the downward z direction. The origin of beam is taken at left end support i.e. at $x = 0$. The material properties for beam used are: $E = 30$ GPa, $\upsilon = 0.2$ and $\rho = 2400$ kg/m³, where E is the Young's modulus, ρ is the density, and υ is the Poisson's ratio of beam material. The boundary conditions associated with simply supported beam as follows:

$M'_{xx} = M_{xx} = w = 0$ at $x = 0$ and $x = L$

$M'_{xx} = M_{xx} = 0 \rightarrow \frac{d\phi}{dx} = \frac{d^2w}{dx^2} = 0$ at $x=0$ and $x=L$

From the general solutions of the beam, expression for w and ϕ as follows:

$$\phi(x) = -\frac{qL^3}{24EI}[4(\tfrac{x}{L})^3 - 6\left(\tfrac{x}{L}\right)^2 + 1] + \frac{qL(A_0 - A_0^2)}{2C_0GA}[(1 - \tfrac{x}{L}) - \frac{2\sinh\lambda(L/2 - x)}{(\lambda L)\cosh(\lambda L/2)}$$

$$w(x) = \frac{qL^4}{24EI}[(\tfrac{x}{L})^4 - 2(\tfrac{x}{L})^3 + \tfrac{x}{L}] - \frac{qA_0^2L^2}{2C_0GA}[\tfrac{x}{L} - (\tfrac{x}{L})^2 - \frac{2}{(\lambda L)^2}\left(1 - \frac{\cosh\lambda(L/2 - x)}{\cosh(\lambda L/2)}\right)] \tag{27}$$

Results are obtained using present theory, other beam theories and exact elasticity solutions given in [16] for axial displacement, transverse displacement, axial stress and transverse stress for two different aspect ratios (S=4 and S=10) of the beam in following non dimensional forms and presented in tables 1-10 and in Fig 2-11 in graphical form.

$$\bar{w}(x) = \frac{10Ebh^3w(x)}{qL^4} \qquad \bar{\sigma}_{xx} = \frac{b\sigma_{xx}(x,z)}{q}$$

$$\bar{\Phi}(x) = \frac{Gb\Phi(x)}{q} \qquad \bar{\tau}_{xz} = \frac{b\tau_{xz}(x,z)}{q}$$

5. Discussion

The results obtained using present theory for bending of simply supported isotropic beam are compared with EBT, FSBT, HSBT, HPSBT and exact elasticity solutions given in [16]. Here the displacements values are obtained using the expressions derived for displacements and stress values are obtained using the constitutive relations.

It is observed that present modification to HPSBT doesn't affect the accuracy of the results and gives exact values as obtained using HPSBT except for the function which represents shear rotation at neutral axis. Results obtained using modified theory almost equal to the values obtained using HSBT. Although present theory overestimates transverse displacement and axial stress compare to exact solution, it is only 0.8% and 0.3% for maximum transverse deflection and axial stress respectively when the aspect ratio is equal to 4. Present theory underestimates the maximum shear stress by 3% and 1.4% for aspect ratio 4 and 10 respectively compare to the exact solution.

Present theory is consistent with the other higher order theories which have been included in the unified beam theory with respect to functions that represent the shear rotation and total rotation at neutral axis. This modification to HPSBT has made the theory more comparable to other higher order theories and it is easier to establish stress resultants and displacement relationships as presented in [10]. Also, the function that is equal to shear rotation at neutral axis in this theory can be replaced in terms of a function that is equal to total rotation of the cross section whereas in HPSBT when the function that represent the shear rotation at neutral axis is replaced with function that represent the total rotation of the cross section, we can't directly get the value for rotation of cross section. It is required to do certain adjustments to get the values for rotation of cross section using HPSBT. This modification would make some tasks much easier like establishing the exact relationship between other beam theories as presented in [10]. Also, this displacement function will be useful in formulating the unified beam element as presented in [11].

Table1. Non dimensional maximum transverse displacement($\bar{w}$),axial stress($\bar{\sigma}_{xx}$), total rotation of cross section($\bar{\Phi}$) and transverse shear stress ($\bar{\tau}_{xz}$) for aspect ratio, S= 4

Model	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\Phi}$	$\bar{\tau}_{xz}$
EBT	1.5625	12.0000	13.3344	-
FSBT	1.7875	12.0000	13.3444	2.4000
Present	1.7872	12.2387	12.7590	2.9111
HPSBT	1.7872	12.2387	-*	2.9111
HSBT	1.7872	12.2351	12.7504	2.9198
Exact	1.7735	12.2000	12.5846	3.0000

*$\bar{\Phi}$ is equal to -7.1414

Table2. Non dimensional maximum transverse displacement($\bar{w}$),axial stress($\bar{\sigma}_{xx}$), total rotation of cross section($\bar{\Phi}$) and transverse shear stress ($\bar{\tau}_{xz}$) for aspect ratio, S= 10

Model	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\Phi}$	$\bar{\tau}_{xz}$
EBT	1.5625	75.0000	208.3500	-
FSBT	1.5985	75.0000	208.3500	6.0000
Present	1.5984	75.2387	206.8882	7.3973
HPSBT	1.5984	75.2387	-*	7.3973
HSBT	1.5984	75.2351	206.8665	7.4198

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Exact	1.5960	75.2000	206.4753	7.5000

*$\bar{\Phi}$ is equal to 156.3421

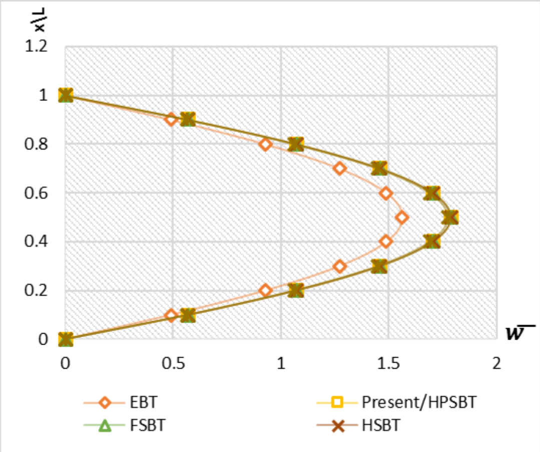

Fig 2. Variations of transverse deflection along the beam(S=4)

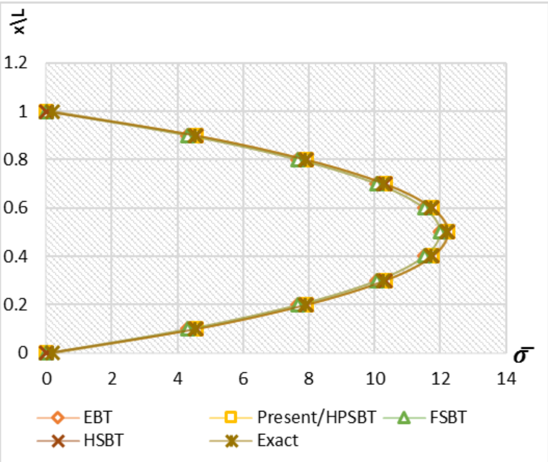

Fig 4. Variations of axial stress along the beam at z=0 (S=4)

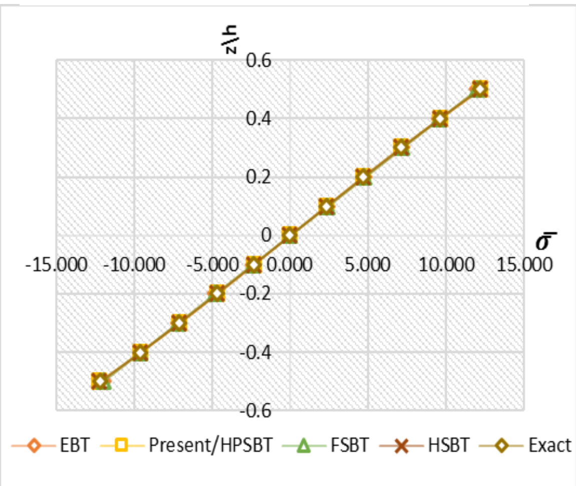

Fig 6. Variation of axial stress across the depth at L= 0 (S=4)

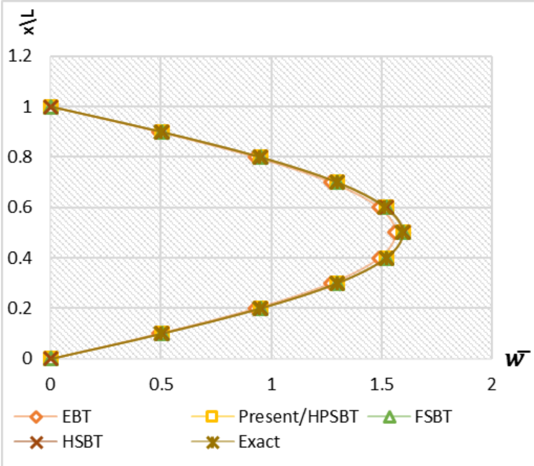

Fig 3. Variations of transverse deflection along the beam(S=4)

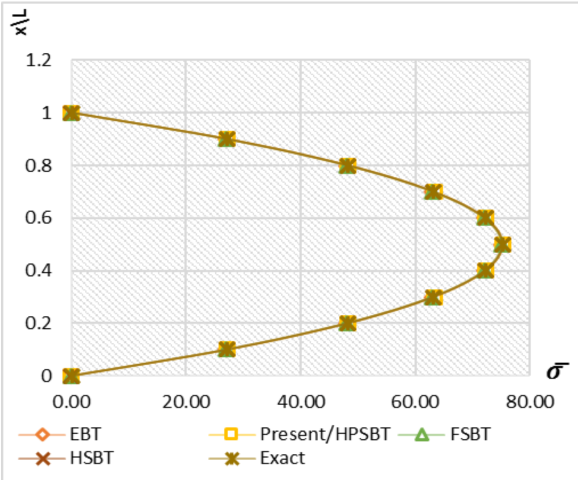

Fig 5. Variations of axial stress along the beam at z=0 (S=10)

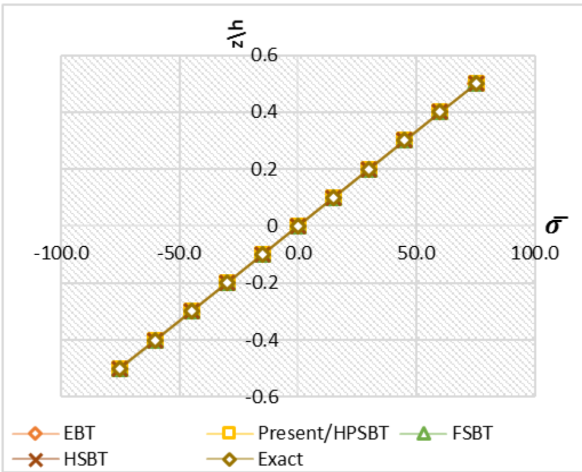

Fig 7. Variation of axial stress across the depth at L= 0 (S=10)

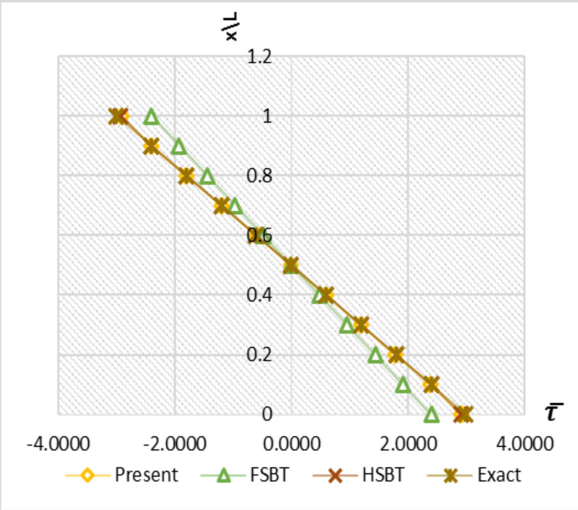

Fig 8. Variation of shear rotation along the beam at z=0 (S=4)

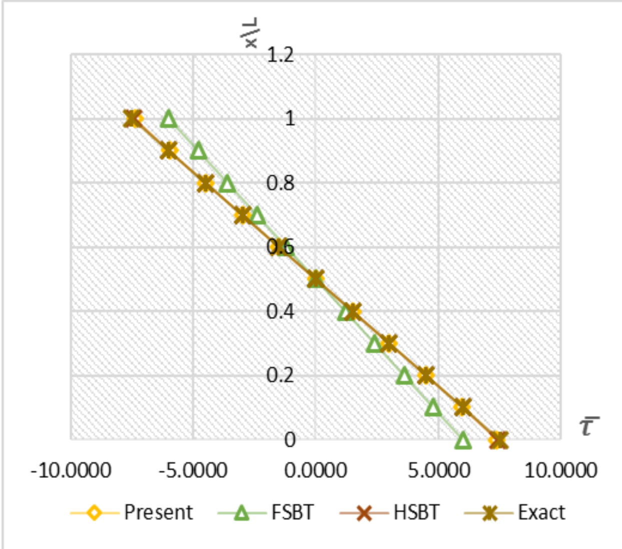

Fig 9. Variation of shear rotation along the beam at z=0 (S=10)

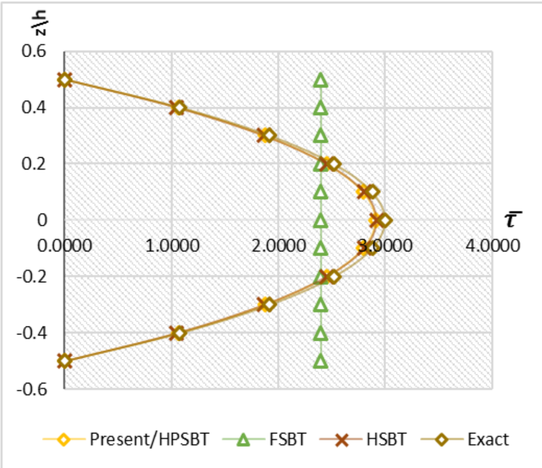

Fig 10. Variation of shear Stress across the depth of beam at L= 0 (S=4)

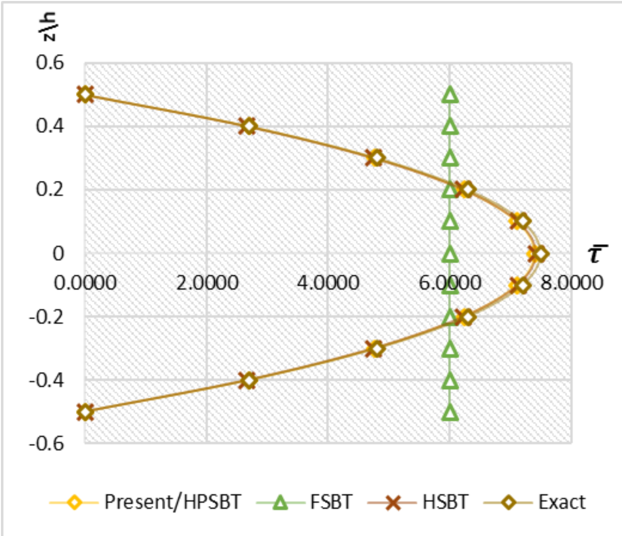

Fig 11. Variation of shear Stress across the depth of beam at L= 0 (S=10)

6. Conclusion

A modified HPSBT has been presented in this paper which has the following features.

Although the displacement field is modified, this theory satisfies zero transverse shear stress boundary conditions on top and bottom surfaces of the beam hence it doesn't need shear correction factor.

The number of unknown variables is same as that of HPSBT.

The axial stress and transverse shear stress can be obtained using the constitutive relations.

The axial stress and transverse shear stress can be obtained using the constitutive relations.

This modified theory gives exactly same values as HPSBT for axial stress, transverse shear stress, transverse displacement and shear strain, except for the function that represents shear rotation at neutral axis.

The present theory gives almost same values compared to HSBT and very close values to exact elastic solution.

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